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Hysteresis and Hemivariational Inequalities: Semilinear Case

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Abstract. In this paper we consider semilinear parabolic boundary value problems having nonsmooth and nonmonotone behaviour and memory effects. The mathematical problem can be formulated and studied by using the notions of hemivariational inequality (based on the generalized gradient in the sense of F.H. Clarke) and the hysteresis operator. We establish two general existence results for such problems. Applications from mechanics illustrate the theory.

Key words: Hysteresis, Hemivariational inequality, Nonconvex superpotential law, Semilinear parabolic boundary value problem

1. Introduction

The theory of hemivariational inequalities has been developed in the last fifteen years in order to fill the gap existing in the variational formulations of B.V.P.s when nonsmooth and generally nonconvex energy functions are involved in the formulations of the problem. For applications and for their mathematical treatment we refer to [18, 21, 23]. When the energy functions become convex, then the hemivariational inequalities become variational inequalities. It is well known that, due to the lack of convexity, compactness arguments must be applied for the mathematical study of the corresponding hemivariational inequalities. Until now eigenvalue problems for hemivariational inequalities have been studied [5, 15, 16] as well as parabolic and hyperbolic problems [6, 12, 13, 23].

Due to the lack of convexity and smoothness the hemivariational inequalities have proved to be an effective tool for the treatment of problems with nonmonotonicity and/or with multivaluedness. In Panagiotopoulos [23, p. 121] it is shown how the hemivariational inequalities can describe loading and unloading processes, whereas in [19] and [21, p. 209] it is explained how a sequence of variational inequalities can describe the classical hysteresis phenomenon with closed and/or open loops.

Parallel to the evolution of the theory of hemivariational inequalities the theory of hysteresis B.V.P.s has been developed. We refer in this respect to the corresponding references [1, 7, 25]. This theory is based on the notion of the hysteresis operator introduced by M.A. Krasnoselskii, which in many cases leads to dynamic variational inequalities and does not even need to appear explicitly [1, p. 5]. In other cases the hysteresis operator is equivalent to a system of infinitely many variational inequalities.

A comparative study of the limitation and of the possibilities of the theory of hemivariational inequalities with the theory of hysteresis is necessary, especially with respect to the possible applications in Mechanics, Engineering and Economics, and it will be the subject of a forthcoming paper. The second author of the present paper, who introduced the notion of hemivariational inequalities, has strong evidence that the two approaches are complementary concerning the treatment of the nonmonotone behaviour of many physical problems and of the phase transition problems: think of a hysteresis operator, which does not have the piecewise monotonicity or the continuity property, and which can describe infinitely many hemivariational inequalities.

In the present paper we will study a parabolic B.V.P. resulting from the superposition of Clarke's generalized gradient, giving rise to a hemivariational inequality, with a continuous hysteresis operator. The organization of the paper is as follows. In Section 2 we recall some basic notations and definitions from the nonsmooth analysis and from the theory of the hysteresis operators. In Section 3 we give the physical motivation for the study of this new type of B.V.P.s. This leads to variational formulations, which are hemivariational inequalities involving a hysteresis operator. In Section 4 we formulate the problem under consideration and state the existence results of Theorems 1 and 2. We have the hysteresis operator in a lower order term, i.e. we do not have time derivatives of the hysteretic term. These types of problems are called semilinear B.V.P.s. with memory, distinguished from quasilinear B.V.P.s with memory, in which the hysteresis operator appears in a higher order term (see this terminology in [25]). The fundamental difference between these two classes is that if the hysteresis operator is in a higher order term, then it has to be piecewise monotone. The difference between Theorem 1 and 2 is that in Theorem 1 we assume that the nonmonotone behaviour obeys the linear growth condition (cf. (J2)), while in Theorem 2 we have only the directional growth condition (cf. (J4)) for nonmonotonicity (which means roughly speaking that it is ultimately increasing). In Section 5 we prove these results. In the proof of Theorem 1 we can apply the standard approach for parabolic problems with hysteresis (see [1, 25]). The proof of Theorem 2 is more involved: the nonsmooth and nonmonotone term has to be regularized and truncated. Moreover, we also need to use the Galerkin method.

2. Preliminaries

2.1. GENERALIZED DERIVATIVES

Let us recall the definitions of the generalized directional derivative and the generalized gradient of F.H. Clarke for a locally Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ from [3]:

(i) The generalized directional derivative of g at ξ in the direction η, denoted $g^\circ(\xi; \eta)$, is defined as follows:

$$g^\circ(\xi; \eta) = \limsup_{\xi' \to \xi, \tau \to 0+} \frac{g(\xi' + \tau\eta) - g(\xi')}{\tau}.$$

(ii) The generalized gradient of g at ξ, denoted $\partial g(\xi)$, is the subset of $\mathbb{R}$ given by

$$\partial g(\xi) = \{\tau \in \mathbb{R} : g^\circ(\xi; \eta) \geqslant \tau\eta \quad \forall \eta \in \mathbb{R}\}.$$

2.2. HYSTERESIS OPERATORS

We recall from [1] some basic concepts of the continuous hysteresis operators, which are needed to formulate the problems under consideration (the continuity means that the input and the output functions of the hysteresis operator are continuous). For extensive treatment and examples of hysteresis operators, like Preisach, Prandtl, elastic–plastic operators, we refer to [1, 7, 25].

Let $[0, T]$ be a given time interval. We denote by $C_{pm}([0, T])$ the set of all continuous and *piecewise monotone* functions on $[0, T]$. A function $v : [0, T] \to \mathbb{R}$ is piecewise monotone if there exists a *monotonicity partition* $\Delta = \{t_i\}_{i=0}^n$, $0 = t_0 < t_1 < \cdots < t_n = T$ such that $v|_{[t_{i-1}, t_i]}$ is monotone for all $i = 1, \ldots n$. The monotonicity partition Δ of v is called the *standard monotonicity partition* of v if the number of the subintervals n is minimal.

We denote by S the set of all strings of real numbers and by S_A the set of all *alternating strings* of real numbers, i.e.

$$S_A = \{(s_0, \ldots, s_n) : (s_{i+1} - s_i)(s_i - s_{i-1}) < 0,\ 1 \leqslant i \leqslant n - 1 \text{ and } n \in \mathbb{N}\}.$$

We define the *restriction operator* $\rho_A : C_{pm}([0, T]) \to S_A$ as follows

$$\rho_A(v) = (v(t_0), \ldots, v(t_n)), \tag{1}$$

where $\{t_i\}_{i=0}^n$ is the standard monotonicity partition of v. Further, the so called *prolongation operator* $\pi_A : S_A \to C_{pm}([0, T])$ maps the string $s = (s_0, \ldots, s_n)$ to the linear interpolate function $v : [0, T] \to \mathbb{R}$ of the points $(\frac{iT}{n}, v(\frac{iT}{n}) \equiv s_i)$, $i = 0, \ldots, n$.

In order to be able to formulate the definition of the *hysteresis operator* we need to introduce the *rate independent functionals*. A functional $\mathcal{H} : C_{pm}([0, T]) \to \mathbb{R}$ is called rate independent if it holds

$$\mathcal{H}[v \circ \phi] = \mathcal{H}[v] \tag{2}$$

for all $v \in C_{pm}([0, T])$ and for all continuous increasing functions $\phi : [0, T] \to \mathbb{R}$ satisfying $\phi(0) = 0$ and $\phi(T) = T$. This implies that only the local extremal values of v are important for the $\mathcal{H}[v]$. Therefore, it is easy to see that the following bijective correspondence [1, Proposition 2.2.5]

$$\mathcal{H} = \tilde{\mathcal{H}} \circ \rho_A, \quad \text{with } \tilde{\mathcal{H}} = \mathcal{H} \circ \pi_A \tag{3}$$

holds between the functions $\tilde{\mathcal{H}} : S_A \to \mathbb{R}$ and the rate independent functionals $\mathcal{H} : C_{pm}([0, T]) \to \mathbb{R}$.

DEFINITION 1. An operator $\mathcal{W} : C_{pm}([0, T]) \to C([0, T])$ is said to be a *hysteresis operator* on $C_{pm}([0, T])$ if there exists a rate independent functional $\mathcal{H}$ called a *generating functional* of $\mathcal{W}$ such that

$$\mathcal{W}[v](t) = \mathcal{H}[v_t], \quad \text{for all } t \in [0, T] \text{ and } v \in C_{pm}([0, T]), \tag{4}$$

in which

$$v_t(\xi) = \begin{cases} v(\xi), & 0 \leqslant \xi \leqslant t, \\ v(t), & t < \xi \leqslant T. \end{cases}$$

Further, an operator $\tilde{\mathcal{W}} : S_A \to S$ is called a *hysteresis operator* on S_A if

$$\tilde{\mathcal{W}}(s) = (\tilde{\mathcal{H}}(s_0), \tilde{\mathcal{H}}(s_0, s_1), \ldots, \tilde{\mathcal{H}}(s)), \quad \text{for all } s = (s_0, \ldots, s_n) \in S_A, \tag{5}$$

where $\tilde{\mathcal{H}} = \mathcal{H} \circ \pi_A$ called a *generating functional* of $\tilde{\mathcal{W}}$ and $\mathcal{H}$ is a rate independent functional on $C_{pm}([0, T])$.

These unique generating functionals $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are often called the *final value mappings* and are denoted by $\mathcal{W}_f$ and $\tilde{\mathcal{W}}_f$, respectively. Due to (3) we also have a bijective correspondence between the hysteresis operators $\mathcal{W}$ defined on $C_{pm}([0, T])$ and $\tilde{\mathcal{W}}$ on S_A. Therefore, in the sequel we can use the same notation $\mathcal{W}$ for both $\mathcal{W}$ and $\tilde{\mathcal{W}}$ and, consequently, the notation $\mathcal{W}_f$ for both $\mathcal{W}_f$ and $\tilde{\mathcal{W}}_f$ without any danger of confusion. In the end, we remark that the hysteresis operators defined on $C_{pm}([0, T])$ can be extended to the set of all continuous functions $C([0, T])$ by using the density of $C_{pm}([0, T])$ in $C([0, T])$ (see [1, 7, 25]).

3. Physical motivation of the present paper and the corresponding problems

The semipermeability problem with hysteretic effects is the pilot problem in this paper. Semipermeability problems were first studied by Duvaut and Lions [4] for

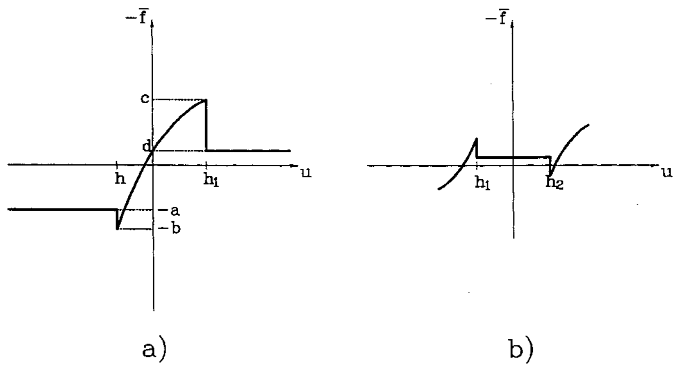

Figure 1. Semipermeability relations without hysteretic effects.

monotone semipermeability conditions. They lead to variational inequalities connected with the Δ-operator and they arise in heat conduction, flow through porous media, and electrostatics. Analogously, they arise in control problems in heat conduction, pressure control in hydraulics etc. The case without monotonicity leads to hemivariational inequalities and was first studied by Panagiotopoulos in [20]. We consider an open bounded connected subset $\Omega \subset \mathbb{R}^3$ referred to a fixed orthogonal Cartesian coordinate system $\mathrm{O}x_1x_2x_3$ and we formulate the equation

$$-\Delta u = f \tag{6}$$

for the stationary problems. On the Lipschitz boundary Γ of Ω we assume that

$$u = 0 \tag{7}$$

and we assume that

$$f = f_1 + f_2 + f_3, \tag{8}$$

where f_2 is given, f_1 is related to u with the relation

$$-f_1 \in \partial j(x, u(x)), \quad \text{in } \Omega_1 \subset \Omega, \tag{9}$$

where j is a locally Lipschitz (i.e. generally nonconvex and nonsmooth) energy function and ∂j denotes its generalized gradient with respect to the second variable. We know [18, 21, 23] that (9) describes, e.g. in the language of heat-conduction problems, the behaviour of a semipermeable membrane of finite thickness occupying a part Ω_1 of Ω, or the behaviour of temperature controller producing a Ω_1 heat in order to regulate the temperature in the interior of Ω. To give an example let us consider Figure 1a.

When the temperature is $u < h$ the region Ω_1 supplies constant heat per unit volume, say a. When $u = h$ heat is supplied for constant temperature until a given value b is reached, the supplied heat–temperature relation follows a parabola as

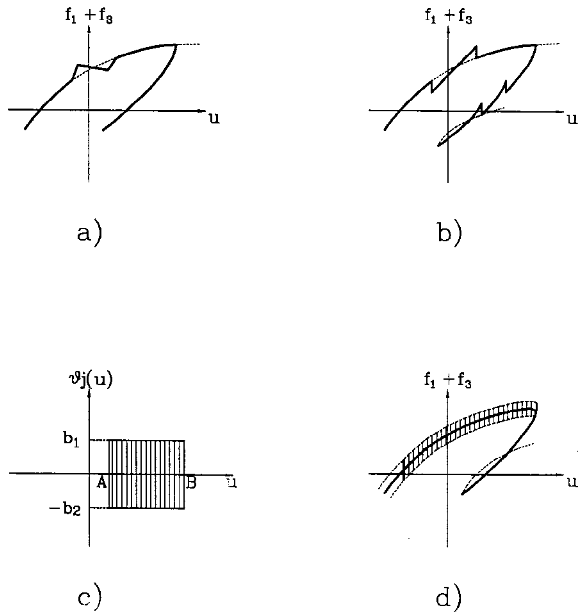

Figure 2. Superposition of nonconvex superpotential laws with hysteresis laws.

in Figure 1a, until the temperature h_1 is reached. We have a change of heat from value $-c$ to $-d$ with the temperature remaining constant, $u = h_1$, and then the heat supply remains constant, whereas the temperature u may increase. Analogously, in Figure 1b a temperature-control problem is depicted in which the temperature is regulated in order to deviate as little as possible from the interval $[h_1, h_2]$.

We assume further that

$$-f_3 = \mathcal{W}_f(w_{-1}(x), u(x); x), \quad \text{in } \Omega_1, \tag{10}$$

where $\mathcal{W}_f$ is the final value mapping of the hysteresis operator $\mathcal{W}$. The function $w_{-1} : \Omega \to \mathbb{R}$ is called the *initial value function* of the hysteresis operator $\mathcal{W}$ defining the initial state of the hysteresis operator. The addition of f_1 and f_3 gives rise to hysteresis mappings of much more general form than the ones treated until now. We can mention that in the one-dimensional case, like here, the hysteresis curves do not need to be piecewise monotone and may contain filled-in jumps (cf. e.g. Figure 2a, b).

REMARK 1. It is assumed that filled-in jumps in mathematical models and laws (cf. e.g. Figures 1a, b, 2a, b) are not attributed to changes to the physical nature of the systems. For instance, in mechanical problems such jumps can cause dynamic effects analogous to impacts.

In the multi-dimensional case of hysteresis operators (cf. [7, p. 151] the multidimensional hysterons) the consideration of sums of hysteresis operators with nonconvex superpotential laws derived by means of the notion of the generalized gradient of Clarke generalizes the theory developed by the researchers on the hysteresis operators. Indeed one can avoid the convexification approach used in [7]. Moreover the direct treatment of the problem without convexification may lead in many cases to results under less stringent assumptions, e.g. by applying the notion of pseudomonotone multivalued mapping, analogous to [18]. Following also the method of Naniewicz related nonconvex star-shaped sets (cf. [18, p. 223]) one can extend the results of [7] to the case of nonconvex star-shaped characteristics (for this notion see [7, p. 156]). Note that generally the notion of nonconvex superpotentials can more easily deal with three-dimensional nonmonotonicities than the notion of hysterons, especially with respect to mechanical laws.

Another possibility offered by the superposition of a nonconvex superpotential law with a hysteresis law is that one can consider “fuzzy hysteretic effects”. We mean (cf. [23, p. 77]) that the $\{f, u\}$ diagram is defined by a set of points lying within a region of given width around the initial graph of the hysteretic law. In this case j must have a special form defined first by Rockafellar ([24, 23, p. 43]). Let l be an open subset of the real line $\mathbb{R}$ and let $\mathcal{M}$ be a measurable subset of l such that for every open and nonempty subset I of l, meas $(I \cap \mathcal{M})$ and meas $(I \cap (l - \mathcal{M}))$ are positive. Let $g(u) = \{b_1,\ \text{if}\ u \in \mathcal{M},\ -b_2,\ \text{if}\ u \notin \mathcal{M}\}$ and $j(u) = \int_0^u g(u^*)\, du^*$. Then j is Lipschitzian and $\partial j(u) = [-b_2, b_1]$ for every $u \in l$, i.e. we have an infinite number of filled-in jumps in l. Hemivariational inequalities with fuzzy superpotentials have been already treated in [18, p. 132].

The aim of the present paper is the study of the following B.V.P. of the parabolic type: The problem (P) is defined as follows

$$\begin{aligned}
&u'(t) + w(t) + Au(t) + \Xi(t) = f(t), \quad \text{a.e. in } (0, T),\\
&w(x, t) = \mathcal{W}[u(x, \cdot); x](t), \quad \forall t \in [0, T], \quad \text{a.e. } x \in \Omega,\\
&\Xi(x, t) \in \partial j(x, u(x, t)), \quad \text{a.e. } (x, t) \in Q_T = \Omega \times (0, T),\\
&u(0) = u_0.
\end{aligned}$$

Since we apply the method of finite differences for the time derivatives in the existence proofs we obtain as a byproduct the existence of the solution for the sequence $\{(\mathrm{P})_E^n\}$ of the corresponding elliptic problems of (P) (of course, some obvious modifications have to be done for the assumptions of $\mathcal{W}$, A and j; cf. Section 5.1 Step I): Let u_i be a solution of $(\mathrm{P})_E^i$, $i = 1, \dots, n-1$, then the problem $(\mathrm{P})_E^n$ is defined by

$$\begin{aligned}
&w_n + Au_n + \Xi_n = f_n\\
&w_n(x) = \mathcal{W}_f(u_0(x), u_1(x), \dots, u_{n-1}(x), u_n(x); x), \quad \text{a.e. } x \in \Omega,\\
&\Xi_n(x) \in \partial j(x, u_n(x)), \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

In the above B.V.P.s we have assumed that $\Omega_1 \equiv \Omega$ for the sake of simplicity.

4. Mathematical formulation of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let V be a Hilbert space such that the imbedding $V \subset H^1(\Omega)$ is dense and continuous. Then $V \subseteq H \equiv L^2(\Omega) \subseteq V^*$ forms an evolution triple. We denote by $\|\cdot\|_V$, $\|\cdot\|_{V^*}$ and $|\cdot|_H$ the norms of V, V^* and H, respectively. The duality pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle_V$ and the inner product in $L^2(\Omega)$ by $(\cdot,\cdot)_H$.

Let $k \in \mathbb{N}$. We suppose that there exists the Galerkin basis $\{\phi_1, \ldots, \phi_k, \ldots\}$ of $C^\infty(\overline{\Omega}) \cap V$ such that $\cup_{k=1}^{\infty} V_k$, $V_k = \{\phi_1, \ldots, \phi_k\}$, is dense in $\tilde{V} \equiv V \cap C(\overline{\Omega})$ in the following sense

$$\forall v \in \tilde{V} \quad \exists\{v_k\}, v_k \in V_k : v_k \to v, \quad \text{in } V \text{ and } C(\overline{\Omega}). \tag{11}$$

Moreover we assume that $V \cap C(\overline{\Omega})$ is dense in V.

For the space-dependent hysteresis operator $\mathcal{W}[\cdot, x]$, which means that the hysteresis operator can vary with the space variable x, we impose the following assumption:

(H) The hysteresis operator $\mathcal{W}[\cdot; x]$ is continuous on $C([0, T])$ for every $x \in \Omega$ and the parametrized final value mapping

$$(s; x) \mapsto \mathcal{W}_f(s; x)$$

is measurable for all $s = (s_0, \ldots, s_n) \in S$ and $n \in \mathbb{N}$ and satisfies

$$|\mathcal{W}_f(s; x)| \leqslant \gamma(x) + c_1 \max_{i=0,\ldots,n} |s_i|, \quad \text{for all } x \in \Omega,\ s \in S \text{ and } n \in \mathbb{N},$$

where $\gamma \in L^2(\Omega)$ and c_1 a positive constant.

REMARK 2. We refer to [1] (e.g. Proposition 2.4.9 and Remark 3.1.1) for examples of Prandtl and Preisach type hysteresis operators and to [7] for examples of hysterons which satisfy the conditions (H). Indeed, those results show that (H) is not a very restrictive condition for the continuous hysteresis operators.

Let A be an operator from V to V^* satisfying:

(A1) The operator A is linear, bounded and symmetric.

(A2) The operator A is coercive, i.e. there exist constants $c_2 > 0$ and $c_3 \geqslant 0$ such that

$$\langle Av, v\rangle_V \geqslant c_2\|v\|_V^2 - c_3|v|_H^2, \quad \forall v \in V.$$

For a function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ we impose the following conditions:

(J1) The function satisfies the Caratheodory type conditions

(i) For all $\xi \in \mathbb{R}$ the function $x \mapsto j(x, \xi)$ is measurable on Ω.

(ii) For almost all $x \in \Omega$ the function $\xi \mapsto j(x,\xi)$ is locally Lipschitz on $\mathbb{R}$.

(J2) Linear growth condition: There exists a positive constant c_4 such that

$$\eta \in \partial j(x,\xi) \Longrightarrow |\eta| \leqslant c_4(1+|\xi|)$$

for a.e. $x \in \Omega$ and each $\xi \in \mathbb{R}$. Moreover, the function $j(\cdot,0) \in L^1(\Omega)$.

(J3) Integrability conditions: There exists a function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\beta(\cdot,r) \in L^2(\Omega)$ for each $r \geqslant 0$,

(ii) If $r' \leqslant r''$ then for almost all $x \in \Omega$

$$\beta(x,r') \leqslant \beta(x,r'')$$

and for almost all $x \in \Omega$

$$|j(x,\xi)-j(x,\eta)| \leqslant \beta(x,r)|\xi-\eta|, \quad \forall \xi,\eta \in B(0,r),\ r \geqslant 0.$$

Moreover, the function $j(\cdot,0) \in L^1(\Omega)$.

(J4) Directional growth condition: There exists a nonnegative function $\alpha_1 : \Omega \to \mathbb{R}$ such that $\alpha_1 \in L^2(\Omega)$ and for almost all $x \in \Omega$

$$j^\circ(x,\xi;-\xi) \leqslant \alpha_1(x)(1+|\xi|) \quad \forall \xi \in \mathbb{R}.$$

REMARK 3. Let us suppose that the conditions (J3) and (J4) hold. Using a similar reasoning as in [18, Remark 5.6] it is possible to show that there exists a nonnegative function $\alpha_2 : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

(i) $\alpha_2(\cdot,r) \in L^2(\Omega)$ for each $r \geqslant 0$.

(ii) If $r' \leqslant r''$ then for almost all $x \in \Omega$

$$\alpha_2(x,r') \leqslant \alpha_2(x,r'')$$

and for almost all $x \in \Omega$

$$j^\circ(x,\xi;\eta-\xi) \leqslant \alpha_2(x,r)(1+|\xi|), \quad \forall \xi \in \mathbb{R}, \quad \eta \in B(0;r), \quad r \geqslant 0. \tag{12}$$

REMARK 4. Let θ be a measurable function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$ such that

$$\gamma(\xi) = \text{ess sup}_{x\in\Omega}|\theta(x,\xi)|$$

belongs to $L^\infty_{loc}(\mathbb{R})$. If the following growth condition

$$\text{ess sup}_{(x,\xi)\in\Omega\times(-\infty,-\bar{\xi})}\theta(x,\xi) \leqslant 0 \leqslant \text{ess inf}_{(x,\xi)\in\Omega\times(\bar{\xi},\infty)}\theta(x,\xi) \tag{13}$$

is satisfied with some positive constant $\bar{\xi}$, a locally Lipschitz function defined by

$$j(x,\xi) = \int_0^{\xi} \theta(x,\eta)\,\mathrm{d}\eta$$

fulfills the assumption (J1), (J3), (J4). The condition (J2) is guaranteed only if θ as a function of ξ obeys a similar linear growth condition.

Further, we state the following assumptions:

(I1) The initial value $u_0 \in V$ and $w_0 \equiv \mathcal{W}_f(w_{-1}(\cdot), u_0(\cdot); \cdot) \in H$.

(I2) The initial value $u_0 \in V \cap L^\infty(\Omega)$ and $w_0 \equiv \mathcal{W}_f(w_{-1}(\cdot), u_0(\cdot); \cdot) \in H$. There exists a sequence $\{u_{0k}\}$, $u_{0k} \in V_k$, satisfying

$$u_{0k} \to u_0, \quad \text{strongly in } H$$

and $w_{0k} \equiv \mathcal{W}_f(w_{-1}(\cdot), u_{0k}(\cdot); \cdot) \in H$. Moreover, $\{u_{0k}\}$ is bounded in V and $L^\infty(\Omega)$.

(F) Let $f \in L^2(0,T;H)$.

The function $w_{-1} : \Omega \to \mathbb{R}$ is the initial value function of the hysteresis operator $\mathcal{W}$ representing the initial state of the hysteresis operator before u_0 or u_{0k} is applied to it at time $t = 0$.

Let us define $Y = H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$. Now we can introduce a *weak formulation* of the problem (P).

DEFINITION 2. A function $u \in Y$ is a solution of the problem (P) if

(i) There exist $w \in L^2(\Omega; C([0,T]))$ and $\Xi \in L^1(Q_T) \cap L^2(0,T;V^*)$ such that

$$\int_0^T (u'(t), v(t))_H\,\mathrm{d}t + \int_0^T (w(t), v(t))_H\,\mathrm{d}t + \int_0^T \langle Au(t), v(t)\rangle_V\,\mathrm{d}t + \int_0^T \langle \Xi(t), v(t)\rangle_V\,\mathrm{d}t = \int_0^T (f(t), v(t))_H\,\mathrm{d}t \quad \forall v \in L^2(0,T;V); \tag{14}$$

$$w(x,t) = \mathcal{W}[u(x,\cdot);x](t), \quad \forall t \in [0,T], \quad \text{a.e. } x \in \Omega; \tag{15}$$

$$\Xi(x,t) \in \partial j(x, u(x,t)) \qquad \text{a.e. } (x,t) \in Q_T. \tag{16}$$

(ii) The function u satisfies the initial condition $u(0) = u_0$.

In this paper we prove the following existence results for (P).

THEOREM 1. *Let hypotheses (H), (A1), (A2), (J1), (J2), (I1), (F) be satisfied. Then the problem (P) has at least one solution. Moreover, the function Ξ belongs to $L^\infty(0,T;H)$.*

THEOREM 2. *Let hypotheses (H), (A1), (A2), (J1), (J3), (J4), (I2), (F) be satisfied. Then the problem (P) has at least one solution.*

5. The proofs of the main theorems

5.1. THE PROOF OF THEOREM 1

We use the standard approach for parabolic problems with hysteresis: time-discretization, a priori estimates and limit procedure (see [1, 25]).

First, we define the semidiscrete problem $(P)_k$ by applying the implicit time-discretization: Let $m \in \mathbb{N}$. We set $k = T/m$ and

$$f_k^n(x) = \frac{1}{k}\int_{(n-1)k}^{nk} f(x,t)\,\mathrm{d}t, \quad \text{for all } n = 1,\dots,m, \tag{17}$$

$$u_k^0(x) = u_0(x). \tag{18}$$

The problem $(P)_k$ is formulated as follows: Find $u_k^n \in V$ and $w_k^n, \Xi_k^n \in H$ for all $n = 1,\dots,m$ such that

$$\frac{u_k^n - u_k^{n-1}}{k} + w_k^n + Au_k^n + \Xi_k^n = f_k^n, \quad \text{in } V^*, \tag{19}$$

$$w_k^n(x) = \mathcal{W}_f(u_k^0(x), u_k^1(x), \dots, u_k^n(x); x), \quad \text{a.e. in } \Omega, \tag{20}$$

$$\Xi_k^n(x) \in \partial j(x, u_k^n(x)), \quad \text{a.e. in } \Omega. \tag{21}$$

Step I: Solvability of $(P)_k$. For each time step $n = 1,\dots,m$ we can rewrite the equation $(P)_k$ as follows: Find $u_k^n \in V$ and $w_k^n, \Xi_k^n \in H$ such that

$$kAu_k^n + u_k^n + kw_k^n + k\Xi_k^n = kf_k^n + u_k^{n-1}, \quad \text{in } V^*, \tag{22}$$

$$w_k^n(x) = \mathcal{W}_f(u_k^0(x), u_k^1(x), \dots, u_k^n(x); x), \quad \text{a.e. in } \Omega, \tag{23}$$

$$\Xi_k^n(x) \in \partial j(x, u_k^n(x)), \quad \text{a.e. in } \Omega. \tag{24}$$

We assume that Problems (22)–(24) are solved for the previous time steps $i = 1,\dots,n-1$. Therefore, the functions $u_k^0,\dots,u_k^{n-1} \in V$ are known.

We use the following result [18, Theorem 4.25] for the static hemivariational inequalities:

THEOREM 3. *Let B be a pseudomonotone operator from V to V^*. Suppose that there exists a function $c : \mathbb{R}_+ \to \mathbb{R}$ with $c(r) \to \infty$ as $r \to \infty$, such that for all $v \in V$, $\langle Bv, v\rangle_V \geqslant c(\|v\|_V)\|v\|_V$, g is an element of V^*, and $\bar{j} : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfills the conditions (J1), (J2) and (J4). Then the hemivariational inequality*

$$Bu + \bar{\Xi} = g \quad \textit{in } V^* \quad \textit{and} \quad \bar{\Xi}(x) \in \partial\bar{j}(x, u(x)), \quad \textit{a.e. in } \Omega$$

has a solution.

Let us first recall the definition of the pseudomonotonicity for single-valued operators from [18, p. 25]. Let B be a mapping from V into V^*. Then B is pseudomonotone if the following hold:

(i) B is bounded;

(ii) If $\{u_i\}$ is a sequence in V converging weakly to u and $\limsup\langle Bu_i, u_i - u\rangle_V \leqslant 0$, then it holds that

$$\liminf\langle Bu_i, u_i - v\rangle_V \geqslant \langle Bu, u - v\rangle_V, \quad \forall v \in V.$$

In order to apply Theorem 3 we define

$$\begin{aligned}
&B := B_1 + B_2,\\
&B_1 v := kAv + \frac{1}{2}v,\\
&(B_2 v)(x) \equiv b(x, v(x)) := k\mathcal{W}_f(u_k^0(x), u_k^1(x), \ldots, u_k^{n-1}(x), v(x); x),\\
&\bar{j}(x, \xi) := kj(x, \xi) + \frac{1}{4}\xi^2,\\
&g := kf_k^n + u_k^{n-1}.
\end{aligned}$$

We assume the time increment k is so small that it satisfies $kc_3 < 1/2$. Then, B_1 is linear and coercive ($\langle B_1 v, v\rangle_V \geqslant c_2\|v\|_V^2$ for all $v \in V$) and, consequently, maximal monotone and $D(B_1) = V$ ($D(B_1) = \{v \in V : B_1(v) \neq \emptyset\}$). Thus, [18, Proposition 2.3] implies that B_1 is pseudomonotone.

Using (H) and the fact that $u_k^0, \ldots, u_k^{n-1}$ belong to V we see that the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the classical Carathéodory conditions and the growth condition

$$|b(x, \xi)| \leqslant k\bar{\gamma}(x) + kC_1|\xi|, \quad \text{for all } (x, \xi) \in \mathbb{R}^N \times \mathbb{R}, \tag{25}$$

where $\bar{\gamma}$ is a nonnegative function from $L^2(\Omega)$ and C_1 a positive constant. Therefore, B_2 is a continuous and bounded Nemyckii operator from $L^2(\Omega)$ to $L^2(\Omega)$ (see e.g. [26, Proposition 26.7]), which, of course, implies that B_2 is pseudomonotone as an operator from V to V^*.

Next we apply the result that the sum of two pseudomonotone operators is pseudomonotone (see e.g. [18, Proposition 2.4]) to B. Further, it is easy to see that B satisfies the coercivity in the sense of Theorem 3 if k is small enough.

Finally, we observe that $\bar{j}$ satisfies (J1), (J2) and (J4) if k is small enough. Thus, all the assumptions of Theorem 3 are satisfied and Problems (22)–(24) have solutions.

Step II: A priori estimates. In derivation of the a priori estimates we apply repeatedly the classical relations:

$$(a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2, \quad \forall a, b \in \mathbb{R}; \tag{26}$$

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon}b^2, \quad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0. \tag{27}$$

We multiply the Equation (19) by $u_k^n - u_k^{n-1}$ and sum it from $n = 1$ to $n = l$, $1 \leqslant l \leqslant m$. Then we estimate the result term by term.

First, we treat the term coming from the hysteresis operator. The use of the Cauchy–Schwartz inequality and (27) yield

$$\begin{aligned}\sum_{n=1}^{l}(w_k^n, u_k^n - u_k^{n-1})_H &\leqslant \left(\sum_{n=1}^{l} k|w_k^n|_H^2\right)^{1/2} \left(\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2\right)^{1/2} \\ &\leqslant C_1(\varepsilon)\sum_{n=1}^{l} k|w_k^n|_H^2 + \varepsilon \sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2\end{aligned} \tag{28}$$

for all $\varepsilon > 0$, where $C_1(\varepsilon) > 0$ is a constant depending only on ε. For estimating $|w_k^n|_H$ in (28) we apply the inequality of (H) giving

$$|w_k^n(x)| \leqslant \gamma(x) + c_1 \sup_{0 \leqslant i \leqslant n} |u_k^i(x)|. \tag{29}$$

Setting

$$v_k^i(x) = |u_0(x)| + \sum_{j=1}^{i} |u_k^j(x) - u_k^{j-1}(x)|, \quad i = 0, \ldots, n \tag{30}$$

and noting that the triangle inequality implies

$$|u_k^i(x)| \leqslant v_k^i(x) \leqslant v_k^{i+1}(x) \leqslant \cdots \leqslant v_k^n(x), \quad i = 0, \ldots, n \tag{31}$$

we can simplify the relation (29) as follows

$$|w_k^n(x)| \leqslant \gamma(x) + c_1 v_k^n(x). \tag{32}$$

Then we substitute (32) into (28) yielding

$$\begin{aligned}\sum_{n=1}^{l}(w_k^n, u_k^n - u_k^{n-1})_H &\leqslant 2C_1(\varepsilon)T|\gamma|_H^2 + 2C_1(\varepsilon)c_1^2 \sum_{n=1}^{l} k|v_k^n|_H^2 \\ &\quad + \varepsilon \sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2.\end{aligned} \tag{33}$$

Next, we show an auxiliary inequality. Indeed, due to an easy calculation we get that

$$\begin{aligned} |v_k^l|_H - |v_k^0|_H &= \sum_{n=1}^{l} \left\{ |v_k^n|_H - |v_k^{n-1}|_H \right\} \leqslant \sum_{n=1}^{l} |v_k^n - v_k^{n-1}|_H \\ &\leqslant T^{\frac{1}{2}} \left(\sum_{n=1}^{l} k \left| \frac{v_k^n - v_k^{n-1}}{k} \right|_H^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{34}$$

By means of a useful relation $|v_k^n - v_k^{n-1}|_H = |u_k^n - u_k^{n-1}|_H$ and the inequality (34) we conclude that

$$\begin{aligned} \sum_{n=1}^{l} \left(\frac{u_k^n - u_k^{n-1}}{k}, u_k^n - u_k^{n-1} \right)_H &= \sum_{n=1}^{l} k \left| \frac{u_k^n - u_k^{n-1}}{k} \right|_H^2 = \sum_{n=1}^{l} k \left| \frac{v_k^n - v_k^{n-1}}{k} \right|_H^2 \\ &\geqslant \frac{1}{2T} |v_k^l|_H^2 - C_1 |u_0|_H^2. \end{aligned} \tag{35}$$

The use of (26), (A1) and (A2) implies

$$\begin{aligned} \sum_{n=1}^{l} \langle Au_k^n, u_k^n - u_k^{n-1} \rangle_V =& \frac{1}{2} \sum_{n=1}^{l} \big(\langle Au_k^n, u_k^n \rangle_V - \langle Au_k^{n-1}, u_k^{n-1} \rangle_V \\ & + \langle A(u_k^n - u_k^{n-1}), u_k^n - u_k^{n-1} \rangle_V \big) \\ =& \frac{1}{2} \langle Au_k^l, u_k^l \rangle_V - \frac{1}{2} \langle Au_0, u_0 \rangle_V \\ & + \frac{1}{2} \sum_{n=1}^{l} \langle A(u_k^n - u_k^{n-1}), u_k^n - u_k^{n-1} \rangle_V \\ & \geqslant \left(\frac{1}{2} c_2 \|u_k^l\|_V^2 + \frac{1}{2} c_2 \sum_{n=1}^{l} \|u_k^n - u_k^{n-1}\|_V^2 \right) \\ & - \frac{1}{2} \langle Au_0, u_0 \rangle_V \\ & + \left(-\frac{1}{2} c_3 |u_k^l|_H^2 - \frac{1}{2} c_3 \sum_{n=1}^{l} |u_k^n - u_k^{n-1}|_H^2 \right). \end{aligned} \tag{36}$$

In (36) we need further estimates for $|u_k^l|_H^2$ and $\sum_{n=1}^{l}|u_k^n - u_k^{n-1}|_H^2$. Applying once again (26) we get

$$\begin{aligned}
&\frac{1}{2}|u_k^l|_H^2 - \frac{1}{2}|u_k^0|_H^2 + \frac{1}{2}\sum_{n=1}^{l}|u_k^n - u_k^{n-1}|_H^2 \\
&\quad = \sum_{n=1}^{l}\frac{1}{2}\left(|u_k^n|_H^2 - |u_k^{n-1}|_H^2 + |u_k^n - u_k^{n-1}|_H^2\right) \\
&\quad = \sum_{n=1}^{l} k\left(\frac{u_k^n - u_k^{n-1}}{k}, u_k^n\right)_H . \qquad (37)
\end{aligned}$$

By rearranging the terms in (37) and using (27) we have

$$\begin{aligned}
&\frac{1}{2}|u_k^l|_H^2 + \frac{1}{2}\sum_{n=1}^{l}|u_k^n - u_k^{n-1}|_H^2 \leqslant \frac{1}{2}|u_0|_H^2 + \varepsilon\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2 \\
&\quad + C_2(\varepsilon)\sum_{n=1}^{l} k|u_k^n|_H^2. \qquad (38)
\end{aligned}$$

After substitution of (38) in (36) we can conclude

$$\begin{aligned}
\sum_{n=1}^{l}\langle Au_k^n, u_k^n - u_k^{n-1}\rangle_V \geqslant &\left(\frac{1}{2}c_2\|u_k^l\|_V^2 + \frac{1}{2}c_2\sum_{n=1}^{l}\|u_k^n - u_k^{n-1}\|_V^2\right) \\
&- c_3\varepsilon\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2 \\
&- c_3C_2(\varepsilon)\sum_{n=1}^{l} k|u_k^n|_H^2 - C_2. \qquad (39)
\end{aligned}$$

Furthermore, the Cauchy–Schwartz inequality and (27) imply

$$\begin{aligned}
\sum_{n=1}^{l}(f_k^n, u_k^n - u_k^{n-1})_H &\leqslant \left(\sum_{n=1}^{l} k|f_k^n|_H^2\right)^{1/2}\left(\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2\right)^{1/2} \\
&\leqslant C_3(\varepsilon) + \varepsilon\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2. \qquad (40)
\end{aligned}$$

Similarly, and taking into account the linear growth condition (J2), we have

$$\sum_{n=1}^{l}(\Xi_k^n, u_k^n - u_k^{n-1})_H \leqslant \left(\sum_{n=1}^{l} k|\Xi_k^n|_H^2\right)^{1/2}\left(\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2\right)^{1/2} \tag{41}$$
$$\leqslant C_4(\varepsilon)\left(1 + \sum_{n=1}^{l} k|u_k^n|_H^2\right) + \varepsilon\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2.$$

Summarizing the estimates (33), (39), (40), (41) and applying the continuity of the imbedding $V \subset H$ we deduce that

$$(1 - C_3\varepsilon)\sum_{n=1}^{l} k\left|\frac{u_k^n - u_k^{n-1}}{k}\right|_H^2 + \left(\frac{1}{2}c_2 - C_5(\varepsilon)k\right)\|u_k^l\|_V^2$$
$$+ \frac{1}{2}c_2\sum_{n=1}^{l}\|u_k^n - u_k^{n-1}\|_V^2$$
$$\leqslant C_6(\varepsilon) + C_7(\varepsilon)\sum_{n=1}^{l-1} k\|u_k^n\|_V^2 + C_8(\varepsilon)\sum_{n=1}^{l} k|v_k^n|_H^2. \tag{42}$$

Finally, as a result of (35) we arrive at

$$\frac{1}{2T}\,(1 - C_3\varepsilon - 2TC_8(\varepsilon)k)\,|v_k^l|_H^2 + \left(\frac{1}{2}c_2 - C_5(\varepsilon)k\right)\|u_k^l\|_V^2$$
$$+ \frac{1}{2}c_2\sum_{n=1}^{l}\|u_k^n - u_k^{n-1}\|_V^2$$
$$\leqslant C_9(\varepsilon) + C_7(\varepsilon)\sum_{n=1}^{l-1} k\|u_k^n\|_V^2 + C_8(\varepsilon)\sum_{n=1}^{l-1} k|v_k^n|_H^2. \tag{43}$$

We first choose ε small enough such that $1 - C_3\varepsilon > 1/2$. This fixes the constants $C_i(\varepsilon)$. Then, we choose k so small that the coefficients $(1 - C_3\varepsilon - 2TC_8(\varepsilon)k)$ and $(1/2c_2 - C_5(\varepsilon)k)$ are positive. After that we can apply the discrete version of Gronwall's lemma implying

$$\max_{1\leqslant n\leqslant m} |v_k^n|_H^2 \leqslant \text{const}, \tag{44}$$
$$\max_{1\leqslant n\leqslant m} \|u_k^n\|_V^2 \leqslant \text{ const}, \tag{45}$$
$$\sum_{n=1}^{m}\|u_k^n - u_k^{n-1}\|_V^2 \leqslant \text{const} \tag{46}$$

for all $0 < k \leqslant k_0$. Further, taking into account (44), (45) we can derive from (42) that

$$\sum_{n=1}^{m} k \left| \frac{u_k^n - u_k^{n-1}}{k} \right|_H^2 \leqslant \text{const}, \tag{47}$$

and from (45) and (J2) it follows easily that

$$\max_{1 \leqslant n \leqslant m} |\Xi_k^n|_H^2 \leqslant \text{ const} \tag{48}$$

for all $0 < k \leqslant k_0$. In the end, we use (32), (44) to see that

$$\max_{1 \leqslant n \leqslant m} |w_k^n|_H^2 \leqslant \text{ const} \tag{49}$$

for all $0 < k \leqslant k_0$.

Step III: Limit procedure. Let us define the piecewise linear interpolate

$$u_k(x, (n+s)k) := s u_k^{n+1}(x) + (1-s) u_k^n(x), \quad s \in (0, 1],$$

and the piecewise constant interpolates

$$\bar{u}_k(x, (n+s)k) := u_k^{n+1}(x), \quad s \in (0, 1],$$
$$\bar{w}_k(x, (n+s)k) := w_k^{n+1}(x), \quad s \in (0, 1],$$
$$\bar{\Xi}_k(x, (n+s)k) := \Xi_k^{n+1}(x), \quad s \in (0, 1],$$
$$\bar{f}_k(x, (n+s)k) := f_k^{n+1}(x), \quad s \in (0, 1],$$

for all $n = 0, \ldots, m-1$. Using these definition it is possible to rewrite the problem $(\mathrm{P})_k$ as follows:

$$u_k'(t) + \bar{w}_k(t) + A\bar{u}_k(t) + \bar{\Xi}_k(t) = \bar{f}_k(t), \quad \text{in } V^*, \tag{50}$$
$$\bar{w}_k(x, t) = \mathcal{W}[\bar{u}_k(x, \cdot); x](t), \quad \text{a.e. in } \Omega, \tag{51}$$
$$\bar{\Xi}_k(x, t) \in \partial j(x, \bar{u}_k(x, t)), \quad \text{a.e. in } \Omega \tag{52}$$

for all $t \in (0, T)$. Due to (45)–(49) we know that

$$\|u_k'\|_{L^2(0,T;H)}, \|u_k\|_{L^\infty(0,T;V)}, \|\bar{u}_k\|_{L^\infty(0,T;V)}, \|\bar{w}_k\|_{L^\infty(0,T;H)}, \|\bar{\Xi}_k\|_{L^\infty(0,T;H)} \tag{53}$$

are bounded for all $0 < k \leqslant k_0$.

Due to the a priori estimates (53) we know that there exist subsequences and limit functions such that

$$u_k \to u, \quad \text{weakly in } H^1(0, T; H) \quad \text{and} \quad \text{weak–}{*} \text{ in } L^\infty(0, T; V), \tag{54}$$
$$\bar{u}_k \to \bar{u}, \quad \text{weak–}{*} \text{ in } L^\infty(0, T; V), \tag{55}$$
$$\bar{w}_k \to w, \quad \text{weak–}{*} \text{ in } L^\infty(0, T; H), \tag{56}$$
$$\bar{\Xi}_k \to \Xi, \quad \text{weak–}{*} \text{ in } L^\infty(0, T; H), \tag{57}$$

as $k \to 0+$. Moreover, from (46) we deduce that

$$\|u_k - \bar{u}_k\|^2_{L^2(0,T;H)} = \sum_{n=1}^{m} \int_{(n-1)k}^{nk} \left(\frac{nk-t}{k}\right)^2 |u_k^n - u_k^{n-1}|_H^2 \,\mathrm{d}t \tag{58}$$

$$\leqslant \sum_{n=1}^{m} k|u_k^n - u_k^{n-1}|_H^2 \leqslant kC_1 \sum_{n=1}^{m} \|u_k^n - u_k^{n-1}\|_V^2 \to 0, \quad \text{as } k \to 0+.$$

As a consequence of this we get that $\bar{u} = u$.

We multiply (50) by $v \in L^2(0,T;V)$ and integrate it over $(0,T)$. Using the convergence results (54)–(57), $\bar{f}_k \to f$ strongly in $L^2(0,T;H)$ and the weak continuity of A we get (14).

In order to prove (15) we need the following compactness result [1, 25],

$$H^1(0,T;H) \cap L^\infty(0,T;V) \text{ is compactly imbedded in } L^2(\Omega; C([0,T])). \tag{59}$$

Thus, recalling (54) we know that

$$u_k \to u, \quad \text{strongly in } L^2(\Omega; C([0,T])) \tag{60}$$

and, consequently,

$$u_k \to u, \quad \text{uniformly in } [0,T] \text{ a.e. in } \Omega. \tag{61}$$

We use the following generalized majorized convergence theorem ([26, Appendix]) for proving that the sequence $\{w_k^*(x,t) = \mathcal{W}[u_k(x,\cdot);x](t)\}$ converges strongly to $w^*(x,t) = \mathcal{W}[u(x,\cdot);x](t)$ in $L^2(\Omega; C([0,T]))$:

$$\lim_{k\to 0+} \int_\Omega g_k(x)\,\mathrm{d}x = \int_\Omega \lim_{k\to 0+} g_k(x)\,\mathrm{d}x \tag{62}$$

if the following conditions are fulfilled:

(i) There exist integrable functions h_k, h satisfying the convergence $h_k(x) \to h(x)$ a.e. in Ω and $\int_\Omega h_k(x)\,\mathrm{d}x \to \int_\Omega h(x)\,\mathrm{d}x$ as $k \to 0+$.

(ii) $|g_k(x)| \leqslant h_k(x)$ a.e. in Ω and $0 < k \leqslant k_0$.

(iii) $\lim_{k\to 0+} g_k(x)$ exists a.e. in Ω.

Indeed: We set

$$g_k(x) = (\sup_{0\leqslant t\leqslant T} |w_k^*(x,t) - w^*(x,t)|)^2,$$

$$h_k(x) = 2(\gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u_k(x,t)|)^2 + 2(\gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u(x,t)|)^2,$$

$$h(x) = 4(\gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u(x,t)|)^2.$$

Due to (H) and (61) we have the following properties

$$\sup_{0\leqslant t\leqslant T} |w_k^*(x,t)| \leqslant \gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u_k(x,t)|, \quad \text{for a.e. } x\in\Omega, \tag{63}$$

$$\sup_{0\leqslant t\leqslant T} |w^*(x,t)| \leqslant \gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u(x,t)|, \quad \text{for a.e. } x\in\Omega, \tag{64}$$

$$w_k^* \to w^*, \quad \text{uniformly in } [0,T] \text{ a.e. in } \Omega. \tag{65}$$

Therefore, the properties (ii), (iii) are easily satisfied. In addition, (60) implies that h_k converges strongly to h in $L^1(\Omega)$ and, consequently, gives (i). Thus, we have proved that w_k^* converges strongly to w^* in $L^2(\Omega; C([0,T]))$. Noting that $\bar{w}_k$ is the piecewise constant interpolate of w_k^* ($\bar{w}_k(x,t) = w_k^*(x,t) = \mathcal{W}[u_k(x,\cdot);x](t)$ as $t = nk$, $n = 1,\dots,m$) it holds also that

$$\sup_{0\leqslant t\leqslant T} |\bar{w}_k(x,t)| \leqslant \gamma(x) + c_1 \sup_{0\leqslant t\leqslant T} |u_k(x,t)|, \quad \text{for a.e. } x\in\Omega. \tag{66}$$

$$\bar{w}_k \to w^*, \quad \text{uniformly in } [0,T] \text{ a.e. in } \Omega. \tag{67}$$

Hence, repeating the previous arguments we see that $\bar{w}_k \to w^*$ in $L^2(\Omega; C([0,T])$ and due to (56) also $w = w^*$.

The last step is to prove (16). By virtue of (57), (58), (60) we have

$$\bar{u}_k \to u, \quad \text{strongly in } L^2(Q_T), \tag{68}$$

$$\bar{\Xi}_k \to \Xi, \quad \text{weakly in } L^2(Q_T). \tag{69}$$

Because of (68) we also have the pointwise convergence of $\bar{u}_k$ to u a.e. in Q_T (by passing to a subsequence, if necessary). Let $\varepsilon > 0$ be arbitrary. Egoroff's theorem implies that there exists $\omega \subset Q_T$ such that meas$(\omega) < \varepsilon$ and $\bar{u}_k$ converges uniformly in $Q_T\backslash\omega$. Thus, $\bar{u}_k, u \in L^\infty(Q_T\backslash\omega)$. Let $\phi \in L^\infty(Q_T)$ be given. Then, due to Fatou's lemma and the upper semicontinuity of the generalized directional derivative, we get

$$\begin{aligned}\int_{Q_T\backslash\omega} \Xi(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \lim_{k\to 0+} \int_{Q_T\backslash\omega} \bar{\Xi}_k(x,t)\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \limsup_{k\to 0+} \int_{Q_T\backslash\omega} j^\circ((x,\bar{u}_k(x,t));\phi(x,t))\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_{Q_T\backslash\omega} \limsup_{k\to 0+} j^\circ((x,\bar{u}_k(x,t));\phi(x,t))\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \int_{Q_T\backslash\omega} j^\circ((x;u(x,t));\phi(x,t))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{70}$$

From this we can conclude

$$\Xi(x,t) \in \partial j(x,u(x,t)), \quad \text{a.e. in } Q_T\backslash\omega. \tag{71}$$

Letting ε tend to zero we get (16). This completes the proof, since the initial condition for u is trivially satisfied.

5.2. THE PROOF OF THEOREM 2

Since we no longer have the linear growth condition (J2), the proof of Theorem 1 has to be modified in many respects. Firstly, the estimate (41) is not valid. Therefore, we have to smooth the function j by using a mollifier and after that to truncate its derivative. Then it is possible to repeat the previous proof for this regularized and truncated problem $(\mathrm{P})_k$. As the regularization and truncation parameter k tends to infinity we can use an elementary differentiation rule (cf. (88)) and the directional growth condition (J4) for controlling this nonmonotone term. Secondly, the proof of Theorem 1 guarantees for the solution of the problem $(\mathrm{P})_k$ only the regularity $H^1(0, T; H) \cap L^\infty(0, T; V)$, not $H^1(0, T; V)$. Thus, we cannot estimate the term $\int_0^T \langle Au_k(t), u_k'(t)\rangle_V \,\mathrm{d}t$ (cf. (84)), which is essential for establishing the required a priori estimates. This difficulty can be removed by working in finite-dimensional Galerkin spaces V_k.

We define the regularized and truncated Galerkin problem $(\mathrm{P})_k$, $k \in \mathbb{N}$. Let ρ be a mollifier such that $\rho \in C_0^\infty((-1, 1))$, $\rho \geqslant 0$ and $\int_{\mathbb{R}} \rho(\eta)\,\mathrm{d}\eta = 1$. We set $\rho_k(\eta) \equiv k\rho(k\eta)$. Then the regularization j_k of j is defined by the convolution

$$j_k(x, \xi) = \int_{\mathbb{R}} \rho_k(\eta) j(x, \xi - \eta)\,\mathrm{d}\eta.$$

On the other hand, the truncation operator of level k is defined as follows:

$$T_k g(\cdot) = \begin{cases} g(\cdot), & |g(\cdot)| \leqslant k; \\ \operatorname{sign}(g(\cdot))k, & |g(\cdot)| > k, \end{cases}$$

where g is a real-valued function.

Now we can formulate $(\mathrm{P})_k$: Find functions $u_k \in Y_k = H^1(0, T; V_k) \cap L^\infty(0, T; V_k)$ and $w_k \in L^2(\Omega; C([0, T]))$ such that

$$u_k'(t) + w_k(t) + Au_k(t) + T_k\partial_2 j_k(\cdot, u_k(t)) = f(t), \quad \text{in } V_k^*, \tag{72}$$

$$w_k(x, t) = \mathcal{W}[u_k(x, \cdot); x](t), \quad \text{a.e. in } \Omega, \tag{73}$$

for almost all $t \in (0, T)$ ($\partial_2 j_k$ denotes the derivative of j_k with respect to the second variable) and

$$u_k(0) = u_{0k}, \tag{74}$$

where $\{u_{0k}\}$ satisfies (I2).

REMARK 5. According to [18, Lemma 5.2] Remark 3 is valid also for $\partial_2 j_k$, $k \in \mathbb{N}$ with a possibly greater increasing nonnegative function $\alpha_2 : \Omega \times \mathbb{R}_+ \to \mathbb{R}$, α_2 independent of k, i.e. particularly for all $k \in \mathbb{N}$ and for almost all $x \in \Omega$

$$\partial_2 j_k(x,\xi)(\eta-\xi) \leqslant \alpha_2(x,r)(1+|\xi|), \quad \forall \xi \in \mathbb{R}, \quad \eta \in B(0;r), \quad r \geqslant 0. \tag{75}$$

It is easy to see also that the truncated function $T_k \partial_2 j_k$ satisfies (75) with the same function α_2 as $\partial_2 j_k$: Indeed, if $\partial_2 j_k(x,\xi) \geqslant 0$ and $(\eta-\xi) \geqslant 0$ we have

$$T_k \partial_2 j_k(x,\xi)(\eta-\xi) \leqslant \partial_2 j_k(x,\xi)(\eta-\xi) \leqslant \alpha_2(x,r)(1+|\xi|)$$

and if $\partial_2 j_k(x,\xi) \geqslant 0$ and $(\eta-\xi) \leqslant 0$

$$\partial_2 j_k(x,\xi)(\eta-\xi) \leqslant T_k \partial_2 j_k(x,\xi)(\eta-\xi) \leqslant 0 \leqslant \alpha_2(x,r)(1+|\xi|).$$

Analogously, one can check the case $\partial_2 j_k(x,\xi) \leqslant 0$.

Step I: Solvability of (P)$_k$. We define a fully discrete approximate problem (P)$_{kh}$ (h is the time increment parameter): Find $u_{kh}^n \in V_k$ and $w_{kh}^n, \Xi_{kh}^n \in H$ for all $n = 1, \ldots, m$ such that

$$\frac{u_{kh}^n - u_{kh}^{n-1}}{h} + w_{kh}^n + Au_{kh}^n + T_k \partial_2 j_k(\cdot, u_{kh}^n) = f_h^n, \quad \text{in } V_k^*, \tag{76}$$

$$w_{kh}^n(x) = \mathcal{W}_f(u_{kh}^0(x), u_{kh}^1(x), \ldots, u_{kh}^n(x); x), \quad \text{a.e. in } \Omega, \tag{77}$$

where $u_{kh}^0 = u_{0k}$ and f_h^n defined by (17).

Next, we repeat Steps I–III of Section 5.1 for the problem (P)$_{kh}$ as h tends to 0 and k is fixed. This gives the existence result for the problem (P)$_k$.

Step II: A priori estimates. We use a continuous analogue of the approach used in Section 5.1. It differs essentially from the discrete one only in how we treat the nonmonotone term.

We set $v = u_k'(s)$ in (P)$_k$ and integrate it over $(0,t)$. First, we define

$$v_k(x,t) = |u_{0k}(x)| + \int_0^t |u_k'(x,s)|\, \mathrm{d}s \tag{78}$$

being a continuous counterpart of (30). Due to an elementary estimation and (78) we have

$$\begin{aligned} u_k(x,r) &= u_{0k}(x) + \int_0^r u_k'(x,s)\, \mathrm{d}s \\ &\leqslant |u_{0k}(x)| + \int_0^t |u_k'(x,s)|\, \mathrm{d}s = v_k(x,t), \quad \forall r \in [0,t]. \end{aligned} \tag{79}$$

Moreover, the condition (H) and (79) implies that

$$w_k(x,t) \leqslant \gamma(x) + c_1 \sup_{0\leqslant s\leqslant t} |u_k(x,s)| \leqslant \gamma(x) + c_1 v_k(x,t). \tag{80}$$

Taking into account (80) we obtain

$$\begin{aligned}\int_0^t (w_k(s), u_k'(s))_H \,\mathrm{d}s &\leqslant \|w_k\|_{L^2(0,t;H)} \|u_k'\|_{L^2(0,t;H)} \\ &\leqslant C_1(\varepsilon)|\gamma|_H^2 + C_1(\varepsilon)\|v_k\|_{L^2(0,t;H)}^2 + \varepsilon\|u_k'\|_{L^2(0,t;H)}^2. \end{aligned}\tag{81}$$

Next, we estimate as in (34)

$$\begin{aligned}|v_k(t)|_H - |v_k(0)|_H &= \int_0^t \frac{d}{\mathrm{d}s}|v_k(s)|_H \,\mathrm{d}s \\ &\leqslant \int_0^t |v_k'(s)|_H \,\mathrm{d}s \leqslant T^{\frac{1}{2}}\|v_k'\|_{L^2(0,t;H)} \end{aligned}\tag{82}$$

and use the relation $v_k'(x,t) = |u_k'(x,t)|$ to deduce

$$\|u_k'\|_{L^2(0,t;H)}^2 = \|v_k'\|_{L^2(0,t;H)}^2 \geqslant \frac{1}{2T}|v_k(t)|_H^2 - C_1|v_k(0)|_H^2. \tag{83}$$

From the conditions (A1), (A2) it follows

$$\begin{aligned}\int_0^t \langle Au_k(s), u_k'(s)\rangle_V \,\mathrm{d}s &= \frac{1}{2}\int_0^t \frac{d}{\mathrm{d}s}\langle Au_k(s), u_k(s)\rangle_V \,\mathrm{d}s \\ &= \frac{1}{2}\langle Au_k(t), u_k(t)\rangle_V - \frac{1}{2}\langle Au_{0k}, u_{0k}\rangle_V \\ &\geqslant \frac{1}{2}c_2\|u_k(t)\|_V^2 - \frac{1}{2}c_3|u_k(t)|_H^2 - \frac{1}{2}\langle Au_{0k}, u_{0k}\rangle_V. \end{aligned}\tag{84}$$

A simple calculation shows that

$$\begin{aligned}\frac{1}{2}|u_k(t)|_H^2 - \frac{1}{2}|u_{0k}|_H^2 &= \frac{1}{2}\int_0^t \frac{d}{\mathrm{d}s}(u_k(s), u_k(s))_H \,\mathrm{d}s = \int_0^t (u_k(s), u_k'(s))_H \,\mathrm{d}s \\ &\leqslant \varepsilon\|u_k'\|_{L^2(0,t;H)}^2 + C_2(\varepsilon)\|u_k\|_{L^2(0,t;H)}^2. \end{aligned}\tag{85}$$

Combining this with (84) we get

$$\begin{aligned}\int_0^t \langle Au_k(s), u_k'(s)\rangle_V \,\mathrm{d}s \geqslant & \frac{1}{2}c_2\|u_k(t)\|_V^2 - \varepsilon c_3\|u_k'\|_{L^2(0,t;H)}^2 \\ & - C_2(\varepsilon)c_3\|u_k\|_{L^2(0,t;H)}^2 - C_2. \end{aligned}\tag{86}$$

Furthermore, it holds

$$\int_0^t (f(s), u_k'(s))_H \,\mathrm{d}t \leqslant C_3(\varepsilon) + \varepsilon\|u_k'\|_{L^2(0,t;H)}^2. \tag{87}$$

We define a truncated function $\bar{v}_k$ of u_k as follows (as a matter of fact $\bar{v}_k(\cdot) = T_1 u_k(\cdot)$):

$$\bar{v}_k(x,t) = \begin{cases} u_k(x,t), & |u_k(x,t)| \leqslant 1; \\ \operatorname{sign}(u_k(x,t)), & |u_k(x,t)| > 1. \end{cases}$$

Using an elementary differentiation rule we can rewrite $\int_0^t (T_k \partial_2 j_k(\cdot, u_k(s)), u_k'(s))_H \, \mathrm{d}s$ in the following form

$$\begin{aligned} &\int_0^t \int_\Omega T_k \partial_2 j_k(x, u_k(x,s)) u_k'(x,s) \, \mathrm{d}x \, \mathrm{d}s \\ &\quad = \int_0^t \int_\Omega \frac{d}{\mathrm{d}s} \int_0^{u_k(x,s)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad = \int_0^t \frac{d}{\mathrm{d}s} \int_\Omega \int_0^{u_k(x,s)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x \, \mathrm{d}s \\ &\quad = \int_\Omega \int_0^{u_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x - \int_\Omega \int_0^{u_{0k}(x)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x. \end{aligned} \tag{88}$$

Due to the definition of $\bar{v}_k$ we have

$$\begin{aligned} \int_\Omega \int_0^{u_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x &= \int_\Omega \int_0^{\bar{v}_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x \\ &\quad + \int_\Omega \int_{\bar{v}_k(x,t)}^{u_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x. \end{aligned} \tag{89}$$

For the first term in (89) it holds

$$\begin{aligned} \Big| \int_\Omega \int_0^{\bar{v}_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x \Big| &\leqslant \int_\Omega \max_{\xi \in [-1,1]} |T_k \partial_2 j_k(x,\xi)| \, \mathrm{d}x \\ &\leqslant \int_\Omega \max_{\xi \in [-1,1]} |\partial_2 j_k(x,\xi)| \, \mathrm{d}x. \end{aligned} \tag{90}$$

By means of (J3) we can estimate as follows:

$$\begin{aligned} |\partial_2 j_k(x,\xi)| &= \Big| \lim_{\eta \to 0} \int_{\mathbb{R}} \rho_k(\tau) \frac{j(x, \xi - \tau + \eta) - j(x, \xi - \tau)}{\eta} \, \mathrm{d}\tau \Big| \\ &\leqslant \lim_{\eta \to 0} \int_{\mathbb{R}} \rho_k(\tau) \Big| \frac{j(x, \xi + \eta - \tau) - j(x, \xi - \tau)}{\eta} \Big| \, \mathrm{d}\tau \leqslant \beta(x, 3), \end{aligned} \tag{91}$$

as $|\xi| \leqslant 1$. Therefore, because of (90), (91) we obtain

$$\Big| \int_\Omega \int_0^{\bar{v}_k(x,t)} T_k \partial_2 j_k(x,\xi) \, \mathrm{d}\xi \, \mathrm{d}x \Big| \leqslant m_N(\Omega)^{\frac{1}{2}} |\beta(3)|_H. \tag{92}$$

We set $\eta = 0$ in (75) implying

$$-\xi T_k \partial_2 j_k(x,\xi) \leqslant \alpha_2(x,0)(1+|\xi|). \tag{93}$$

If $\xi \geqslant 1$, (93) yields

$$T_k \partial_2 j_k(x,\xi) \geqslant -2\alpha_2(x,0). \tag{94}$$

Similarly, if $\xi \leqslant -1$ we can deduce that

$$T_k \partial_2 j_k(x,\xi) \leqslant 2\alpha_2(x,0). \tag{95}$$

Because of (94), (95) we note that

$$\begin{aligned}\int_\Omega \int_{\bar{v}_k(x,t)}^{u_k(x,t)} T_k \partial_2 j_k(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x &\geqslant -2\int_\Omega |u_k(x,t)|\alpha_2(x,0)\,\mathrm{d}x \\ &\geqslant -2|\alpha_2(0)|_H |u_k(t)|_H \\ &\geqslant -\varepsilon\|u_k(t)\|_V^2 - C_4(\varepsilon).\end{aligned} \tag{96}$$

Recalling that $\{u_{0k}\}$ is bounded in $L^\infty(\Omega)$, i.e. $\|u_{0k}\|_{L^\infty(\Omega)} \leqslant C_3$ we get as in (90), (91)

$$\left|\int_\Omega \int_0^{u_0(x)} T_k \partial_2 j_k(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\right| \leqslant m_N(\Omega)^{\frac{1}{2}}|\beta(C_3+2)|_H. \tag{97}$$

Taking into account (92), (96), (97) we arrive at

$$\int_0^t (T_k \partial_2 j_k(\cdot, u_k(s)), u_k'(s))_H\,\mathrm{d}s \geqslant -\varepsilon\|u_k(t)\|_V^2 - C_5(\varepsilon). \tag{98}$$

Then combining the estimates (81), (86), (87), (98) we conclude that for all $t \in (0,T]$ it holds

$$\begin{aligned}&(1 - C_4\varepsilon)\|u_k'\|^2_{L^2(0,t;H)} + \left(\frac{1}{2}c_2 - \varepsilon\right)\|u_k(t)\|_V^2 \\ &\quad\leqslant C_6(\varepsilon) + C_7(\varepsilon)\|u_k\|^2_{L^2(0,t;V)} + C_1(\varepsilon)\|v_k\|^2_{L^2(0,t;H)}.\end{aligned} \tag{99}$$

Finally, we employ the relation (83) and the continuity of the imbedding $V \subset H$ implying

$$\begin{aligned}&\frac{1}{2T}(1 - C_4\varepsilon)|v_k(t)|_H^2 + \left(\frac{1}{2}c_2 - \varepsilon\right)\|u_k(t)\|_V^2 \\ &\quad\leqslant C_8(\varepsilon) + C_7(\varepsilon)\|u_k\|^2_{L^2(0,t;V)} + C_1(\varepsilon)\|v_k\|^2_{L^2(0,t;H)}.\end{aligned} \tag{100}$$

We now choose ε such that the coefficients of $|v_k(t)|_H^2$ and $\|u_k(t)\|_V^2$ are greater than zero. This, together with Gronwall's lemma, guarantees that

$$\|u_k\|_{L^\infty(0,T;V)}, \|v_k\|_{L^\infty(0,T;H)} \leqslant \text{const} \tag{101}$$

for all $k \in \mathbb{N}$. Moreover, we infer from (80), (99), (101) easily

$$\|w_k\|_{L^\infty(0,T;H)}, \|u_k'\|_{L^2(0,T;H)} \leqslant \text{const} \tag{102}$$

for all $k \in \mathbb{N}$.

Finally, we establish the weak precompactness of the sequence $\{T_k \partial_2 j_k(u_k)\}$ in $L^1(Q_T)$. According to the Dunford–Pettis theorem it is sufficient to prove that for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that

$$\int_\omega |T_k \partial_2 j_k(x, u_k(x,t))| \, \mathrm{d}x \, \mathrm{d}t < \varepsilon \tag{103}$$

for all $\omega \subset Q_T$ and $m_{N+1}(\omega) < \delta(\varepsilon)$.

Applying (75) we see that for any $r > 0$ ($\eta = \pm r$)

$$\pm r T_k \partial_2 j_k(x, u_k(x,t)) \leqslant T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) + \alpha_2(x,r)(1 + |u_k(x,t)|) \tag{104}$$

for a.e. (x,t) in Q_T. Thus,

$$\begin{aligned}\int_\omega |T_k \partial_2 j_k(x, u_k(x,t))| \, \mathrm{d}x \, \mathrm{d}t \leqslant & \frac{1}{r} \int_\omega T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ & + \frac{1}{r} \|\bar{\alpha}_2(r)\|_{L^2(\omega)} (m_{N+1}(\omega)^{\frac{1}{2}} + \|u_k\|_{L^2(\omega)}),\end{aligned} \tag{105}$$

where a function $\bar{\alpha}_2 : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is defined by $\bar{\alpha}_2(x,t,r) = \alpha_2(x,r)$ $\forall t \in [0,T]$. On the other hand, the substitution $\eta = 0$ in (75) yields

$$0 \leqslant T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) + \alpha_2(x,0)(1 + |u_k(x,t)|), \tag{106}$$

for a.e. (x,t) in Q_T. Hence,

$$\begin{aligned}\int_\omega T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t \leqslant & \int_{Q_T} T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ & + \int_{Q_T} \alpha_2(x,0)(1 + |u_k(x,t)|) \, \mathrm{d}x \, \mathrm{d}t \\ \leqslant & \int_{Q_T} T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ & + \|\bar{\alpha}_2(0)\|_{L^2(Q_T)} (m_{N+1}(Q_T)^{\frac{1}{2}} + \|u_k\|_{L^2(Q_T)}) \\ \leqslant & \int_{Q_T} T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t + C_5,\end{aligned} \tag{107}$$

in which we have used (101). Taking into account (A1), (72), (101), (102) we can deduce

$$\begin{aligned}\int_{Q_T} T_k \partial_2 j_k(x, u_k(x,t)) u_k(x,t) \, \mathrm{d}x \, \mathrm{d}t \leqslant & C_6 \|u_k\|^2_{L^2(0,T;V)} \\ & + (\|u_k'\|_{L^2(0,T;H)} + \|w_k\|_{L^2(0,T;H)} + \|f\|_{L^2(0,T;H)}) \|u_k\|_{L^2(0,T;H)} \leqslant C_7.\end{aligned} \tag{108}$$

Combining the above inequalities (105), (107), (108) we find that

$$\int_\omega |T_k \partial_2 j_k(x, u_k(x, s))| \, dx \, ds \leqslant \frac{C_8}{r} + C_9 \|\bar{\alpha}_2(r)\|_{L^2(\omega)}. \tag{109}$$

Let $\varepsilon > 0$ be given. First, we choose $r > 0$ big enough such that

$$\frac{C_8}{r} < \frac{\varepsilon}{2}, \tag{110}$$

and next the parameter $\delta(\varepsilon)$ small enough such that

$$C_9 \|\bar{\alpha}_2(r)\|_{L^2(\omega)} < \frac{\varepsilon}{2} \tag{111}$$

for all $\omega \subset Q_T$ satisfying $m_{N+1}(\omega) < \delta(\varepsilon)$. This is possible, because for any $r > 0$ $\alpha_2(r) \in L^2(\Omega)$ implies that $\bar{\alpha}_2(r) \in L^2(Q_T)$. As a result of (109)–(111) we have established the validity of the Dunford–Pettis's criterion for the weak precompactness of $\{T_k \partial_2 j_k(u_k)\}$ in $L^1(Q_T)$.

Step III: Limit Procedure. Because of the a priori estimates (101)–(103) we know that there exist subsequences and limit functions such that

$$u_k \to u, \quad \text{weakly in } H^1(0, T; H) \quad \text{and} \quad \text{weak–}* \text{ in } L^\infty(0, T; V), \tag{112}$$

$$w_k \to w, \quad \text{weak–}* \text{ in } L^\infty(0, T; H), \tag{113}$$

$$T_k \partial_2 j_k(u_k) \to \Xi, \quad \text{weakly in } L^1(Q_T), \tag{114}$$

as $k \to \infty$. Further, thanks to (59) we also have

$$u_k \to u, \qquad \text{strongly in } L^2(\Omega; C([0, T])). \tag{115}$$

Because of (72) it holds that

$$\begin{aligned} &\int_0^T (u_k'(t), v(t))_H \, dt + \int_0^T (w_k(t), v(t))_H \, dt + \int_0^T \langle A(t) u_k(t), v(t) \rangle_V \, dt \\ &\quad + \int_0^T (T_k \partial_2 j_k(\cdot, u_k(t)), v(t))_H \, dt \\ &\quad = \int_0^T (f(t), v(t))_H \, dt, \quad \forall v \in C(0, T; V_k). \end{aligned} \tag{116}$$

By means of the convergence results (112)–(114) we can take the limit of (116) as $k \to \infty$ implying

$$\begin{aligned} &\int_0^T (u'(t), v(t))_H \, dt + \int_0^T (w(t), v(t))_H \, dt + \int_0^T \langle A(t) u(t), v(t) \rangle_V \, dt \\ &\quad + \int_0^T (\Xi(t), v(t))_H \, dt = \int_0^T (f(t), v(t))_H \, dt, \quad \forall v \in C(0, T; V_k). \end{aligned} \tag{117}$$

Recalling (11) we know that (117) is valid for all $v \in C([0,T];\tilde{V})$. Therefore, we have that

$$\int_0^T (\Xi(t), v(t))_H \,\mathrm{d}t = \int_0^T (f(t) - u'(t) - w(t), v(t))_H \,\mathrm{d}t + \int_0^T \langle Au(t), v(t)\rangle_V \,\mathrm{d}t, \quad \forall v \in C([0,T];\tilde{V}). \tag{118}$$

Then, the density of $C([0,T];\tilde{V})$ in $L^2(0,T;V)$ implies that $\Xi \in L^2(0,T;V^*)$, and, consequently, (14) is satisfied.

Using similar arguments as in Section 5.1 one can confirm the validity of (15). Further, the initial condition $u(0) = u_0$ is an easy consequence of (I2) and the convergence of $\{u_k\}$ to u in $L^2(\Omega; C([0,T]))$.

In order to complete the proof we have to show

$$\Xi(x,t) \in \partial j(x, u(x,t)), \quad \text{a.e. } (x,t) \in Q_T. \tag{119}$$

Repeating the reasoning in Section 5.1 we get that for any $\delta > 0$ there exists $\omega_1 \subset Q_T$, $m_{N+1}(\omega_1) < \delta/2$, such that u_k converges uniformly to u in $Q_T \backslash \omega_1$, and, moreover

$$\|u_k\|_{L^\infty(Q_T\backslash\omega_1)}, \|u\|_{L^\infty(Q_T\backslash\omega_1)} \leqslant C_1, \quad \forall k \geqslant k_0 \tag{120}$$

for some $k_0 \in \mathbb{N}$ and C_1 a positive constant. The definition of the regularization j_k and (J3) permit us to estimate

$$|\partial_2 j_k(x,\xi)| = \left| \lim_{\theta\to 0} \int_{\mathbb{R}} \rho_k(\tau) \frac{j(x,\xi-\tau+\theta) - j(x,\xi-\tau)}{\theta} \,\mathrm{d}\tau \right| \leqslant \beta(x, |\xi|+2), \quad \text{for a.e. in } \Omega. \tag{121}$$

By (J3) we know that $\beta(\cdot, C_1+2) \in L^2(\Omega)$. Thus, there exists $k_0' \in \mathbb{N}$ and a set $\omega_2 \subset Q_T$ such that

$$|\beta(x, C_1+2)| \leqslant k_0', \quad \forall x \text{ s.t. } (x,t) \in Q_T\backslash\omega_2 \quad \text{and} \quad m_{N+1}(\omega_2) < \frac{\delta}{2}. \tag{122}$$

Let us take $k \geqslant \max(k_0, k_0')$. Then

$$T_k \partial_2 j_k(x, u_k(x,t)) = \partial_2 j_k(x, u(x,t)), \quad \text{in } Q_T\backslash(\omega_1 \cup \omega_2). \tag{123}$$

Let $v \in L^\infty(Q_T)$ be given. Using the definition of the generalized directional derivative, Fatou's lemma and (120)–(123), we get

$$\begin{aligned}
&\int_{Q_T\setminus(\omega_1\cup\omega_2)} j^\circ(x,u(x,t);v(x,t))\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{Q_T\setminus(\omega_1\cup\omega_2)} \limsup_{\substack{\theta(x,t)\to 0+\\ \eta(x,t)\to 0}}\\
&\qquad\times\frac{j(x,u(x,t)+\eta(x,t)+v(x,t)\theta(x,t))-j(x,u(x,t)+\eta(x,t))}{\theta(x,t)}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant\int_{Q_T\setminus(\omega_1\cup\omega_2)} \limsup_{\substack{\theta\to 0+\\ k\to\infty}}\int_{\mathbb{R}}\rho_k(\tau)\\
&\qquad\times\frac{j(x,u_k(x,t)-\tau+v(x,t)\theta)-j(x,u_k(x,t)-\tau)}{\theta}\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant\int_{Q_T\setminus(\omega_1\cup\omega_2)} \limsup_{k\to\infty}\partial_2 j_k(x,u_k(x,t))v(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\geqslant\limsup_{k\to\infty}\int_{Q_T\setminus(\omega_1\cup\omega_2)}\partial_2 j_k(x,u_k(x,t))v(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (124)
\end{aligned}$$

By virtue of the convergence $T_k\partial_2 j_k(u_k)\to\Xi$ weakly in $L^1(Q_T)$ and (123) we obtain

$$\int_{Q_T\setminus(\omega_1\cup\omega_2)} j^\circ(x,u(x,t);v(x,t))\,\mathrm{d}x\,\mathrm{d}t\geqslant\int_{Q_T\setminus(\omega_1\cup\omega_2)}\Xi(x,t)v(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (125)$$

Recalling the definition of the generalized gradient the inequality (125) gives

$$\Xi(x,t)\in\partial j(x,u(x,t)),\quad \text{a.e. in}\quad Q_T\setminus(\omega_1\cup\omega_2). \qquad (126)$$

Letting $\delta\to 0+$ implies (119). This completes the proof.

6. Conclusions

We have developed the mathematical theory for problems having nonsmooth, nonmonotone behaviour, and memory effects. More precisely, we considered semilinear parabolic B.V.P.s containing a continuous scalar hysteresis operator and a nonmonotone term expressed by means of the generalized gradient. The advantage of this formulation is that we could treat more general problems: the studied relations (e.g. mechanical laws) could contain nonmonotone discontinuities.

We proved two general existence theorems for such problems. Basically the proofs were based on the standard approach for parabolic problems with hysteresis: time–discretization, a priori estimates and limit procedure, i.e the proofs were constructive. Therefore, we see that in possible numerical realizations the key problem is to solve the static hemivariational inequality at each time step; and if the static problem has a potential, the original problem can be reduced to a sequence of problems of finding substationary points of nonconvex nonsmooth energy functionals (we refer, for example, to [10, 11] for the practical realization).

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